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Note

Parsons graphs of matrices on $\mathscr{Z}_{p^n}$ ☆

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Let R be a finite commutative ring with q elements, d an even integer, and $SL_d(R)$ the special linear group on R of dimension d. For any b in R, let $T_b(d,q)$ denote the following graph:

(1) $V = V(T_b(d,q)) = SL_d(R)$, that is the collection of all the $d \times d$ matrices A over R for which $\det(A) = 1$.

(2) $E = E(T_b(d,q))$ is the collection of all the pairs (A,B) of elements of V for which $\det(A - B) = b$.

When $R = GF(q)$ is a finite field with q elements, Zaks [2] called $T_b(d,q)$ a Parsons graph, and proposed the following conjecture:

Conjecture. Every Parsons graph, except for $T_1(2,2)$, is connected.

In this paper, we will generalize the concept of a Parsons graph on $GF(q)$ to that on $\mathscr{Z}_{p^n}$ (p a prime), called here the Parsons graph also, and discuss the connectivity of $T_b(d,p^n)$. Our main results are the following two theorems.

Theorem 1. *When $R = \mathscr{Z}_{p^n}$, every Parsons graph, except for $T_{2k+1}(2,2^n), k = 0,1,\ldots, 2^{n-1}-1$, and $T_{3k+2}(2,3^n), k = 0,1,\ldots,3^{n-1}-1$, is connected.*

Theorem 2. *When $R = GF(q)$, every Parsons graph, except for $T_1(2,2)$ and $T_2(2,3)$, is connected.*

We prove them through several lemmas.

First, we observe that $\mathscr{Z}_{p^n}$ is the disjoint union of units and zero divisors, and u is a unit iff $(u,p) = 1$. $SL_d(\mathscr{Z}_{p^n})$ is generated by elements of the form $T_{ij}(t) = I + tE_{ij}$, since $\mathscr{Z}_{p^n}$ is a local ring (see [1, Ch. I]). For any zero divisor z, there exist units u_1 and u_2, such that $z = u_1 + u_2$, so that $T_{ij}(z) = T_{ij}(u_1)T_{ij}(u_2)$, and $SL_d(\mathscr{Z}_{p^n})$ is generated by $\{T_{ij}(u) \mid u \text{ a unit}\}$.

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Therefore, we get the following.

Lemma 1. *When $R = \mathscr{Z}_{p^n}$, $T_b(d, p^n)$ is connected iff for every unit u, and $i \neq j$, $T_{ij}(u)$ is connected to I.*

Lemma 2. *For any even integer integer $d > 2, T_b(d, p^n)$ is connected.*

Proof. Let

$$B_1 = \begin{pmatrix} 1-b & 1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ -b & 0 & \dots & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & \dots & 0 & -b \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 1 & 1-b \end{pmatrix}.$$

It is easy to see that $(I,B_1),(I,B_2) \in E = E(T_b(d,p^n))$. Also for any nonzero element u, if $(i,j) \neq (d,1),(d,2)$, and $(i,i+1)$, where $i = 1,\dots,d-1$, then $(T_{ij}(u),B_1) \in E$. If $(i,j) = (d,1)$ or $(d,2)$, or $(i,i+1)$, then $(T_{ij}(u),B_2) \in E$. Thus any $T_{ij}(u)$, with $u \neq 0$, is connected to I, and therefore, by Lemma 1, $T_b(d,p^n)$ is connected. □

For the case $d = 2$, like the one treated in [2], it is easy to prove the following.

Lemma 3. *For*

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in SL_2(\mathscr{Z}_{p^n}),$$

$(X,I) \in E$ *iff* $\operatorname{tr}(X) = 2 - b$, *and* $(X,A) \in E$ *iff* $\operatorname{tr}(X) = 2 - b + ax_3$.

Lemma 4. *With the exceptions of $p = 2$ and b is a unit, and $p = 3$ and $b \equiv 2 \pmod 3$, for any $b \in \mathscr{Z}_{p^n}$, there always exists a unit $v \in \mathscr{Z}_{p^n}$, such that $v^2 + b$ is a unit.*

Proof. When b is a zero divisor, we may take $v = 1$. When b is a unit, if $p = 2$, then for every unit v, $v^2 + b$ is always a zero divisor; if $p = 3$, we have $v^2 \equiv 1 \pmod 3$ for any unit $v \in \mathscr{Z}_{p^n}$, hence $v^2 + b$ is a unit iff $b \equiv 1 \pmod 3$; if $p > 3$, then there always exists a unit v, such that $v^2 + b$ is a unit. □

Lemma 5. *With the exceptions of $p = 2$ and b is a unit, and $p = 3$ and $b \equiv 2 \pmod 3), T_b(2, p^n)$ is a connected graph.*

Proof. By Lemma 4, there exists a unit v, such that $w = v^2 + b$ is unit. Let u be a unit in $\mathscr{Z}_{p^n}$, and

$$X = \begin{pmatrix} 0 & -uvw^{-1} \\ w(uv)^{-1} & 2-b \end{pmatrix}, \quad Y = \begin{pmatrix} v & * \\ w(uv)^{-1} & 2-b+bv^{-1} \end{pmatrix} \in SL_2(\mathscr{Z}_{p^n}).$$

By Lemma 3, $(X,I),(Y,I) \in E$, and $\det(Y - X) = b$, i.e. $(X,Y) \in E$. Hence A is connected to I. Similarly, I is connected to A^t, the transpose of A. $\square$

Remark. Consider Lemmas 2 and 5, we can see that the proofs also hold for the case when $R = GF(p^n)$, and in this case the exceptional conditions are $p^n = 2$, $b = 1$ and $p^n = 3$, $b = 2$. By direct computations, we see that $T_1(2,2)$ and $T_2(2,3)$ have 2, and 3 components, respectively. This completes the proof of Theorem 2. $\square$

We return to the proof of Theorem 1.

Lemma 6. *The graphs* $T_{2k+1}(2,2^n), k = 0,1,\ldots,2^{n-1} - 1$, *and the graphs* $T_{3k+2}(2,3^n)$, $k = 0,1,\ldots,3^{n-1} - 1$, *are not connected.*

Proof. We see that , when $n = 1$, $\mathscr{Z}_2 = GF(2)$, $\mathscr{Z}_3 = GF(3)$; if $T_{2k+1}(2,2^n)$ or $T_{3k+1}(2,3^n)$ is a connected graph, then by the computations modul 2 or 3, we get that $T_1(2,2)$ or $T_2(2,3)$ is a connected graph, a contradiction to Theorem 2. This completes the proof of Theorem 1. $\square$

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References

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[2] J. Zaks, Parsons graphs of matrices, Discrete Math. 78 (1989) 187–193.